Kuwait University — MATH 102 — July 9, 2011

Math. Dept. — First Midterm Test — Duration 90 minutes

Calculators and mobile telephones are not allowed.

1. (1+3+2 pts) Let $f(x) = (\ln(x-1)+1)^3$.

a) Find the domain of f.

b) Show that f^{-1} exist, and find the domain of f^{-1}.

c) Compute $f^{-1}(x)$.

2. (2 pts each)

a) Find the exact value of $\sec(\tan^{-1}(\frac{5}{2}))$.

b) Solve the equation for x, $1 + \log_2(x-1) = \log_2(x)$.

3. (2 pts) Use logarithmic differentiation to find $\frac{dy}{dx}$ if

$$y = \left(\frac{1+\tan^{-1}(x^2)}{e^x\sqrt{cos(x)}}\right)^x.$$

4. (3 pts each) Evaluate the following integrals.

a) $\displaystyle\int \frac{\tanh x}{\sqrt{\cosh^2 x - 16}}dx$

b) $\displaystyle\int \frac{dx}{(1+e^{-x})\ln(1+e^x)}$

c) $\displaystyle\int \tanh(\ln(x))dx$

5. (2 pts each) Find the following limits.

a) $\displaystyle\lim_{x\to 0+}(\sec x + x)^{\frac{1}{x}}$.

b) $\displaystyle\lim_{x\to 0+}\frac{x\sinh x}{1-\cosh x}$.

Solution

1. $f(x) = (1+\ln(x-1))^3$

(a) $x-1>0 \;\Rightarrow x>1 \;\Rightarrow\; D_f =]1,\infty[$

(b) $f'(x) = \dfrac{3(1+\ln(x-1))^2}{x-1} > 0 \;\; on \;\; x>1 \;\Rightarrow\; f\uparrow on \;\; x>1 \;\Rightarrow\; f \; is \; (1-1) \;\Rightarrow\; f^{-1} exists$

$D_{f^{-1}} = R_f \;=\;]\,f(1^+), f(\infty)\,[\;=\;]-\infty, \infty[$

$[\;\; f(1^+) = \lim_{x\to 1^+} (1+\ln(x-1))^3 = (1+\ln 0^+)^3 = -\infty \;\;\&\;\; f(\infty) = \lim_{x\to\infty} (1+\ln(x-1))^3 = (1+\ln\infty)^3 = \infty \;]$

(c) $y = f(x) = (1+\ln(x-1))^3 \Rightarrow \sqrt[3]{y} - 1 = \ln(x-1) \;\Rightarrow\; x-1 = e^{\sqrt[3]{y}-1} \;\Rightarrow\; f^{-1}(x) = 1 + e^{\sqrt[3]{x}-1}$.

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2. (a) $\sec(\tan^{-1}(5/2)) = \sec x = \sqrt{29}/2$. $\;\left[\, x = \tan^{-1}(5/2) \;\Rightarrow\; \tan x = 5/2 \,\right]$

(b) $1+\log_2(x-1) = \log_2 x \;\Rightarrow\; 1+\ln(x-1)/\ln 2 = \ln x/\ln 2 \;\Rightarrow\; \ln 2 = \ln x - \ln(x-1) = \ln\dfrac{x}{x-1}$

$\Rightarrow\; 2 = \dfrac{x}{x-1} \;\Rightarrow\; \dfrac{1}{2} = \dfrac{x-1}{x} = 1 - \dfrac{1}{x} \;\Rightarrow\; x = 2.$

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3. $y = \left[\dfrac{1+\tan^{-1} x^2}{e^x\sqrt{\cos x}}\right]^x \;\Rightarrow\; \ln y = x\,\{\ln(1+\tan^{-1} x^2) - x - 1/2\,\ln(\cos x)\}$

$\Rightarrow\; y' = \left\{ \left(\ln(1+\tan^{-1} x^2) - x - 1/2\,\cos x \right) + x\left(\dfrac{2x}{(1+x^4)(1+\tan^{-1} x^2)} - 1 + \dfrac{\tan x}{2} \right) \right\} \left[\dfrac{1+\tan^{-1} x^2}{e^x\sqrt{\cos x}}\right]^x$

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4. (a) $\displaystyle\int \frac{\tanh x}{\sqrt{\cosh^2 x - 16}}\,dx = \int \frac{\sinh x}{\cosh x\sqrt{\cosh^2 x - 4^2}}\,dx = \frac{1}{4}\sec^{-1}\left(\frac{\cosh x}{4}\right) + c$

(b) $\displaystyle\int \frac{dx}{(1+e^{-x})\ln(1+e^x)} = \int\frac{dt}{t} = \ln\left|\ln(1+e^x)\right| + c\,.$ $\quad \left\{ t = \ln(1+e^x) \;\Rightarrow\; dt = \dfrac{e^x dx}{1+e^x} = \dfrac{dx}{(1+e^{-x})} \right\}$

(c) $\displaystyle\int \tanh(\ln x)\,dx = \int \frac{x^2-1}{x^2+1}\,dx = \int \left(1 - \frac{2}{x^2+1}\right) dx = x - 2\tan^{-1} x + c$

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5. (a) $\displaystyle\lim_{x\to 0^+} (x+\sec x)^{1/x} = e \qquad \left\{ \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(x+\sec x)}{x} = \lim_{x\to 0^+} \frac{1+\sec x\tan x}{x+\sec x} = 1 \right\}$

(b) $\displaystyle\lim_{x\to 0^+} \frac{x\sinh x}{1-\cosh x} = \lim_{x\to 0^+} \frac{\sinh x + x\cosh x}{-\sinh x} = \lim_{x\to 0^+} \frac{2\cosh x + x\sinh x}{\cosh x} = -2.$
